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Lagrange and Least-Squares Polynomials as Limits of Linear Combinations of Iterates of Bernstein and Durrmeyer Polynomials

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The linear combination of iterates $1-(1-P_n)^M$ of Bernstein and Durrmeyer operators of a fixed degree n is considered for increasing order of iteration M. The resulting sequence of polynomials is shown to converge to the Lagrange interpolating polynomial for the Bernstein operators, and the least-squares approximating polynomial for the Durrmeyer operators.

INTRODUCTION

In [5, 6], the linear combination of iterates $1-(1-P_n)^M$ of elements of a sequence P_n of simultaneous approximation operators was considered as a way to obtain a more rapidly converging sequence of approximants. (There, the combination was identified as the iterated Boolean sum of the operator.) For operators satisfying appropriate error estimates, the modified sequence was shown to give accelerated convergence for smooth functions. The technique was applied in the univariate case to the Bernstein, Stancu, and Durrmeyer operators, for which explicit estimates on the order of convergence were given. A number of authors have considered this particular combination of iterates as a means for accelerating convergence; see [5] or [6] for references.

In the above work, the order M of iteration was held constant as the degree n of the approximants became infinite. In this paper we reverse the roles, and study the sequence of approximants generated if the degree is held constant while the order of iteration becomes infinite. For the Bernstein operators, the sequence of polynomials generated is shown to converge to the Lagrange polynomial of degree n interpolating at equidistant points on [0, 1]. For the Durrmeyer operators, the sequence converges to the least-squares polynomial approximant on [0, 1].

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In the sequel, π_n will represent the space of polynomials of degree $\leqslant n$, $C^p([0,1])$ the space of functions with continuous pth derivative on $[0,1]$, and $L([0,1])$ the space of functions (Lebesgue) integrable on $[0,1]$. $R([0,1])$ will denote the set of real-valued functions defined throughout $[0,1]$ (but not necessarily uniformly bounded).

1. The Bernstein Polynomials

Let $B_n\colon R([0,1]) \to \pi_n([0,1])$ be the Bernstein operators defined by

$$B_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f(k/n).$$

Note that each B_n maps π_r into π_r for $r \leqslant n$ (see, e.g., [1]). For fixed n, we consider the sequence $\{1-(1-B_n)^M\}_{M=1}^{\infty}$.

Theorem 1. *For fixed $n \geqslant 1$ and $f \in R([0,1])$, the sequence $\{1-(1-B_n)^M\}_{M=1}^{\infty}$ converges uniformly on $[0,1]$ as $M \to \infty$ to $L_n(f;x)$, the Lagrange polynomial of f of degree n interpolating at the equidistant nodes $x=0, 1/n, 2/n, ..., n/n$.*

Proof. We begin by showing that the sequence is Cauchy in the uniform norm. For $f \in R([0,1])$ and $M, S \geqslant 1$, we have

$$\begin{aligned}(1-B_n)^M &- (1-B_n)^{M+S}\\ &= (1-B_n)^M B_n + (1-B_n)^{M+1} B_n + \cdots + (1-B_n)^{M+S-1} B_n,\end{aligned}$$

or

$$\begin{aligned}&\|(1-(1-B_n)^M)(f) - (1-(1-B_n)^{M+S})(f)\|_\infty\\ &\qquad = \left\| \sum_{k=0}^{S-1} (1-B_n)^k (1-B_n)^M (p) \right\|_\infty,\end{aligned}$$

where $p(x) = B_n(f;x)$ is a polynomial of degree $\leqslant n$. Kelisky and Rivlin [3] showed that the operator $B_n(f;x)$, as a linear operator from $\pi_n([0,1]) \to \pi_n([0,1])$, has eigenvalues

$$\lambda_j = \begin{cases} 1, & j=0,\\ (n)_j/n^j, & 1 \leqslant j \leqslant n,\end{cases} \qquad \text{where} \quad (a)_r = a\cdot(a-1)\cdots(a-r+1),$$

with each corresponding eigenfunction $p_j(x)$ being a monic polynomial of exact degree j. We can therefore write the polynomial $p(x)$ in terms of this basis of eigenfunctions,

$$p(x) = a_0 + a_1 p_1(x) + a_2 p_2(x) + \cdots + a_n p_n(x).$$

Then

$$\begin{aligned}(1-B_n)^M\,(p;x)\\ &= a_0(1-\lambda_0)^M + a_1(1-\lambda_1)^M\,p_1(x) + \cdots + a_n(1-\lambda_n)^M\,p_n(x),\end{aligned}$$

or

$$\begin{aligned}&\left\|\sum_{k=0}^{S-1}(1-B_n)^k\,(1-B_n)^M\,(p)\right\|_\infty\\ &\quad= \|a_0\Sigma_0 + a_1\Sigma_1 p_1(x) + \cdots + a_n\Sigma_n p_n(x)\|_\infty,\\ &\quad\leqslant |a_0|\left[\frac{(1-\lambda_0)^M}{\lambda_0}\right] + |a_1|\left[\frac{(1-\lambda_1)^M}{\lambda_1}\right]K_1 + \cdots + |a_n|\left[\frac{(1-\lambda_n)^M}{\lambda_n}\right]K_n,\end{aligned}$$

where $\Sigma_j = \sum_{k=0}^{S-1}(1-\lambda_j)^k\,(1-\lambda_j)^M$ and $K_j = \|p_j(x)\|_\infty$. Since $0 \leqslant (1-\lambda_j) < 1$ for all j, it is clear that the sequence $\{1-(1-B_n)^M\}_{M=1}^{\infty}$ is uniformly Cauchy.

Since the sequence is Cauchy, it converges to some $g \in C([0,1])$; indeed, since each element of the sequence is a polynomial of degree $\leqslant n$ (fixed), the same will be true of g. For a given $f \in R([0,1])$, let $g \in \pi_n([0,1])$ be this limiting polynomial:

$$g = \lim_{M\to\infty}(1-(1-B_n)^M)(f).$$

Then

$$g - f = \lim_{M\to\infty}(1-B_n)^M\,(f),$$

or

$$(1-B_n)(g-f) = g-f,$$

whence

$$B_n(g-f) = 0.$$

But $B_n(h) = 0$ iff $h(k/n) = 0$, $k = 0, 1, ..., n$; thus

$$g(k/n) = f(k/n), \qquad k = 0, 1, ..., n,$$

i.e., g is the nth degree Lagrange polynomial of f, interpolating at the equidistant points $x = 0, 1/n, 2/n, ..., n/n$. ▮

2. The Durrmeyer Polynomials

The Durrmeyer operators $D_n: L([0,1]) \to \pi_n([0,1])$ are defined by

$$D_n(f; x) = (n+1) \sum_{k=0}^{n} p_{n,k}(x) \int_0^1 p_{n,k}(t) f(t)\, dt,$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ for $0 \leqslant k \leqslant n$. The approximation properties of these operators were studied by Derriennic in [2]; note in particular that D_n maps π_r into π_r for $r \leqslant n$. Once again, for fixed n, we consider the sequence of combinations of iterates $\{1-(1-D_n)^M\}_{M=1}^{\infty}$.

Theorem 2. *For fixed $n \geqslant 1$ and $f \in L([0,1])$, the sequence $\{1-(1-D_n)^M\}_{M=1}^{\infty}$ converges uniformly as $M \to \infty$ to $Q_n(f; x)$, the orthogonal projection of f onto $\pi_n([0,1])$.*

Note. Orthogonality is with respect to the usual inner product $\langle f, g\rangle = \int_0^1 f(x)\, g(x)\, dx$.

Proof. Derriennic [2] proved that the eigenvalues of D_n are

$$\lambda_j = \begin{cases} 1, & j = 0, \\ (n)_j/(n+j+1)_j, & 1 \leqslant j \leqslant n, \end{cases}$$

with each corresponding eigenfunction $p_j(x)$ being the Legendre orthogonal polynomial on $[0,1]$ of degree j. Because $0 < \lambda_j \leqslant 1$ for all j (and we have a basis of eigenfunctions for $\pi_n([0,1])$), the proof that the sequence $\{1-(1-D_n)^M\}_{M=1}^{\infty}$ is Cauchy and hence converges as $M \to \infty$ follows exactly that of Theorem 1.

To determine the limit function, we can proceed as before. Since each element of the sequence is a polynomial of degree $\leqslant n$ (fixed), the same will be true of the limit. Letting $g \in \pi_n$ be this limiting polynomial, we have $g = \lim_{M\to\infty} (1-(1-D_n)^M)(f)$; then as before,

$$D_n(g-f) = 0.$$

But

$$D_n(h) = 0 \qquad \text{iff} \quad \int_0^1 h(x)\, x^k (1-x)^{n-k}\, dx = 0, \qquad k = 0, 1, ..., n;$$

since the functions $x^k(1-x)^{n-k}$, $k = 0, 1, ..., n$, form a basis for π_n, the above is equivalent to the case in which the first n moments of h are 0:

$$D_n(h) = 0 \qquad \text{iff} \quad \int_0^1 h(x)\, x^k\, dx = 0, \qquad k = 0, 1, ..., n;$$

i.e., h is orthogonal to π_n. Thus, since the error $g-f$ is orthogonal to π_n, the polynomial g is the projection of f onto π_n (see, e.g., [4]); i.e., g is the least-squares approximant to f from π_n. ∎

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